General Extended Mean Value Theorem. Suppose $f(x)$ and its derivatives $f'(x), f''(x), \ldots, f^{(n-1)}(x)$ of order one through $n-1$ are continuous on $a \le x \le b$, and $f^{(n)}(x)$ exists for $a < x < b$. If

$$\begin{aligned} F(x) &= f(x) - f(a) - (x-a)f'(a) \\ &\quad - \frac{(x-a)^2 f''(a)}{2!} - \cdots \\ &\quad - \frac{(x-a)^{n-1} f^{(n-1)}(a)}{(n-1)!} - K(x-a)^n, \end{aligned}$$

where K is chosen so that $F(b) = 0$, show that

(a) $F(a) = F(b) = 0$,

(b) $F'(a) = F''(a) = \cdots = F^{(n-1)}(a) = 0$,

(c) there exist numbers $c_1, c_2, c_3, \ldots, c_n$ such that

$$a < c_n < c_{n-1} < \cdots < c_2 < c_1 < b$$

and such that

$$\begin{aligned} F'(c_1) = 0 &= F''(c_2) \\ &= F'''(c_3) = \cdots = F^{(n-1)}(c_{n-1}) \\ &= F^{(n)}(c_n). \end{aligned}$$

(d) Hence, deduce that

$$K = \frac{f^{(n)}(c_n)}{n!}$$

for c_n as in (c); or, in other words, since $F(b) = 0$,

$$\begin{aligned} f(b) = f(a) &+ f'(a)(b-a) \\ &+ \frac{f''(a)}{2!}(b-a)^2 + \cdots \\ &+ \frac{f^{(n-1)}(a)}{(n-1)!}(b-a)^{n-1} \\ &+ \frac{f^{(n)}(c_n)}{n!}(b-a)^n \end{aligned}$$

for some c_n, $a < c_n < b$. [*Amer. Math. Monthly*, Vol. 60 (1953), p. 415, James Wolfe.]